

# Conservation Laws and Cosmological Perturbations in Curved Universes

Jean-Philippe Uzan<sup>1</sup>, Nathalie Deruelle<sup>1,2</sup> and Neil Turok<sup>2</sup>

<sup>1</sup> Département d'Astrophysique Relativiste et de Cosmologie  
UPR 176 du Centre National de la Recherche Scientifique  
Observatoire de Paris, 92195 Meudon, France

<sup>2</sup> Department of Applied Mathematics and Theoretical Physics  
University of Cambridge, Silver Street  
Cambridge CB3 9EW, England

February 7, 2008

## Abstract

When working in synchronous gauges, pseudo-tensor conservation laws are often used to set the initial conditions for cosmological scalar perturbations, when those are generated by topological defects which suddenly appear in an up to then perfectly homogeneous and isotropic universe. However those conservation laws are restricted to spatially flat ( $K = 0$ ) Friedmann-Lemaître spacetimes. In this paper, we first show that in fact they implement a matching condition between the pre- and post- transition eras and, in doing so, we are able to generalize them and set the initial conditions for all  $K$ . Finally, in the long wavelength limit, we encode them into a vector conservation law having a well-defined geometrical meaning.

**Pacs :** 98.80.Hw, 98.70.Vc, 04.20.Cv

**Ref. PRD :** DD6306

## 1 Introduction

Topological defects act in cosmology as “active” sources for the metric and matter perturbations in that the linear differential equations for the evolution of these perturbations are inhomogeneous (in contrast with what happens in inflationary scenarios). Thus the general solution of the evolution equation for each perturbation mode is the sum of a particular solution of the complete equation and the general solution of the homogeneous equation. It depends on integration constants that have to be fixed by the physics which gave rise to the defects.

One has thus to take into account the fact that the defects appear at a phase transition in a previously strictly homogeneous and isotropic universe. For scales much larger than

the Hubble radius at that time, which are the scales of interest today, the phase transition appears as instantaneous (in the following we shall use indifferently the words “scales” or “wavelengths”). A way to fix the initial conditions is therefore to find linear combinations of the perturbation variables which are constant in time and force them to vanish since they were strictly zero before the phase transition. It has been shown [1] that, in the newtonian gauge, the initial conditions are completely fixed by these matching conditions.

In synchronous gauges, for flat ( $K = 0$ ) Friedmann-Lemaître spacetimes, and for “scalar” perturbations (see [2] for the decomposition of perturbations into scalar, vector and tensor modes), the initial conditions are usually fixed, at least partially (see e.g. [3], [4], [5], [6])), by means of a “conserved pseudo-tensor”  $\tau_{\mu\nu}$  [3], more precisely by the constraint “ $\tau_{00} = 0$ ” (the equation of conservation  $\partial_0 \tau_{00} = \partial_k \tau_{0k} \simeq 0$  on large scales then ensures that this constraint is conserved during the evolution [3]).

In this article, we first compare and contrast the implementation of the initial conditions for long wavelength scalar perturbations in both gauges (& 2 and & 3). To do so we consider a toy model, namely coherent defects in a radiation dominated, spatially flat, universe. We stress some of the Charybdes and Scyllas of studying its perturbations in synchronous gauges, and show how the “pseudo-tensor”  $\tau_{\mu\nu}$  defined in [3] partially fixes the initial conditions. We then interpret the condition “ $\tau_{00} = 0$ ” as a matching condition between the pre- and post- transition eras. Finally, we show how the newtonian gauge simplifies the resolution of the evolution equations. In & 4, because we had related it to a matching condition in the flat case, we can construct a generalized “conserved pseudo-tensor” in synchronous gauges for  $K \neq 0$  Friedmann-Lemaître spacetimes. Finally, in & 5, we formalize the definition of this “pseudo-tensor” by means of a well defined vector conservation law [7] and give it a precise geometrical meaning.

## 2 Scalar perturbations in synchronous gauges

The line element of a perturbed Friedmann-Lemaître spacetime reads, when the perturbations are scalar and in synchronous gauges :

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + \left\{ \left(1 + \frac{1}{3}h\right)\gamma_{ij} + 2(D_i D_j - \frac{1}{3}\gamma_{ij}\Delta)E \right\} dx^i dx^j \right] \quad (1)$$

where  $\eta$  is conformal time,  $a(\eta)$  the background scale factor ;  $\gamma_{ij}$  is the three-metric of the constant time maximally symmetric surfaces in the coordinate system  $x^i$  ( $i = 1, 2, 3$ ) ;  $D_i$  is the associated covariant derivative,  $\Delta \equiv D_i D^i$  (latin indices are raised by means of the inverse metric  $\gamma^{ij}$ ) ; finally  $h$  and  $E$  are two “small” functions of space and time.

Infinitesimal coordinate transformations which preserve the synchronicity of the gauge can be performed. They are defined by :

$$\eta \rightarrow \eta + T \quad , \quad x^k \rightarrow x^k + \partial^k L \quad (2)$$

where  $T$  and  $L$  are two first order functions of  $\eta$  and  $x^i$  such that

$$T' + \mathcal{H}T = 0 \quad \text{and} \quad L' - T = 0 \quad \Longleftrightarrow \quad T = \frac{e_0}{a} \quad \text{and} \quad L = e_1 + e_0 \int \frac{d\eta}{a} \quad (3)$$

where a prime denotes a derivative with respect to conformal time, where  $\mathcal{H} \equiv a'/a$  and where  $e_0$  and  $e_1$  are functions of space only. The two scalar metric perturbations  $h$  and  $E$  then transform as

$$h \rightarrow h + 6\mathcal{H}T + 2\Delta L, \quad E \rightarrow E + L \quad (4)$$

with  $T$  and  $L$  given by (3).

The energy-momentum tensor of the matter content of this perturbed universe can be written as :

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} + \Theta_{\mu\nu}. \quad (5)$$

$\bar{T}_{\mu\nu}$  ( $\mu, \nu = 0, 1, 2, 3$ ) is the energy-momentum tensor of the homogeneous and isotropic background of radiation and dust ;  $\delta T_{\mu\nu}$  is its perturbation : in synchronous gauges, its scalar components can be expressed in terms of two scalar quantities,  $\delta \equiv \delta\rho/\rho$ , the density contrast, and  $v$ , the velocity perturbation, as (see e.g. [1]) :

$$\delta T_{00} = \rho a^2 \delta, \quad \delta T_{0i} = -\rho a^2 (1 + \omega) D_i v, \quad \delta T_{ij} = P a^2 \left[ \left( \frac{h}{3} + \Gamma + \frac{c_s^2}{\omega} \delta \right) \gamma_{ij} + 2D_i D_j E \right]. \quad (6)$$

$\rho$  and  $P$  are the density and pressure of the background fluid,  $\omega \equiv P/\rho$ ,  $c_s^2 \equiv P'/\rho'$  and  $P\Gamma \equiv \delta p - c_s^2 \delta \rho$ .

Finally  $\Theta_{\mu\nu}$  is the energy-momentum tensor of the topological defects. We suppose that it is a small perturbation which does not contribute to the background. We decompose its scalar components as :

$$\Theta_{00} = \rho^s, \quad \Theta_{0i} = -D_i v^s, \quad \Theta_{ij} = \gamma_{ij} \left( P^s - \frac{1}{3} \Delta \Pi^s \right) + D_i D_j \Pi^s. \quad (7)$$

The four source functions  $\rho^s, P^s, v^s, \Pi^s$  will be discussed later.

The energy and momentum conservation equations for  $\Theta_{\mu\nu}$  then read :

$$\rho^{s'} + \mathcal{H}(\rho^s + 3P^s) + \Delta v^s = 0 \quad (8)$$

$$v^{s'} + 2\mathcal{H}v^s + P^s + \frac{2}{3}\Delta \Pi^s = 0. \quad (9)$$

The conservation equations for the fluid are, at linear order (see e.g. [8]) :

$$\delta' + \frac{1}{2}(1 + \omega)h' = -(1 + \omega)\Delta v \quad (10)$$

$$v' + \mathcal{H}(1 - 3c_s^2)v = -\frac{c_s^2}{1 + \omega}\delta. \quad (11)$$

As for the linearized Einstein equations they can be cast under the form (see e.g. [8] or [1]) :

$$-\left(\frac{1}{3}\Delta + K\right)h^- + \mathcal{H}h' = \kappa(a^2\rho\delta + \rho^s) \quad (12)$$

$$\left(\frac{1}{3}\Delta + K\right)h^{-'} - Kh' = \kappa\left[a^2\rho(1 + \omega)\Delta v + \Delta v^s\right] \quad (13)$$

$$h'' + \mathcal{H}h' = -\kappa\left[a^2\rho(1 + 3c_s^2)\delta + (\rho^s + 3P^s)\right] \quad (14)$$

$$(h - h^-)'' + 2\mathcal{H}(h - h^-)' - \frac{1}{3}\Delta h^- = -2\kappa\Delta\Pi^s \quad (15)$$

where  $\kappa \equiv 8\pi G$ ,  $G$  being Newton's constant, and where the perturbation  $h^-$  is defined as

$$h^- \equiv h - 2\Delta E. \quad (16)$$

Equations (8-15) are eight equations for the four unknowns  $h, h^-$  (or  $E$ ),  $\delta$  and  $v$  : the source functions  $\rho^s, P^s, v^s, \Pi^s$ , subject to the constraints (8-9) being known, equations (10-15) must give  $h, h^-$  (or  $E$ ),  $\delta$  and  $v$ , two of these equations being redundant. The four Einstein equations for scalar perturbations (12-15) include two constraints on the four variables so that the general solution depends on two physical degrees of freedom which must be fixed by the physics of the problem.

To be complete, let us recall the Friedmann equations for the background

$$\kappa a^2 \rho = 3(\mathcal{H}^2 + K), \quad \kappa a^2 (\rho + 3P) = -6\mathcal{H}'. \quad (17)$$

### 3 Synchronous gauges inside out

#### 3.1 A toy model : coherent defects in a flat, radiation dominated, universe

To show, on a simple example, how the above system of equations can be solved, we consider here a radiation dominated universe with flat,  $K = 0$ , spatial sections. Hence,  $\omega = c_s^2 = 1/3$ ,  $\Gamma = 0$  and, from (17),  $a(\eta) \propto \eta$ . Moreover we suppose that we are deep enough in the radiation era for the perturbations of interest today to be all larger than the comoving Hubble radius ( $1/\mathcal{H}$ ). Finally we assume that, in that large scale (or long wavelength) limit, the variables describing the defects take the form

$$\rho^s = \eta^{-1/2} A_1, \quad P^s = \eta^{-1/2} A_2, \quad v^s = -\eta^{1/2} A_3, \quad \Pi^s = \eta^{3/2} A_4 \quad (18)$$

where  $A_i$  are four functions (or, rather, four random variables) of space varying on scales larger than the Hubble radius. Provided that the defect network is coherent, uncorrelated on scales larger than the Hubble radius and scales with the background (the precise definition of these properties is given in e.g. [1]), it can be modelled by (18), (see e.g. [4], [10]). Now, the precise energy-momentum tensor of the network of topological defects depends on the kind of defects considered and can be obtained only by heavy numerical simulations (see e.g. [9]). It turns out that, generically, the network is not perfectly coherent so that  $\Delta\Pi^s$ , and not  $\Pi^s$  itself, varies on scales larger than the Hubble radius. However such incoherent sources can, in principle, be decomposed into a sum of coherent sources described by (18) [13].

The conservation equations (8-9) first impose that, in the long wavelength limit (where we neglect the gradient terms)

$$A_1 = -6A_2 \quad \text{and} \quad A_3 = \frac{2}{5}A_2. \quad (19)$$

Then equation (10) and (14) can be rewritten, in the same approximation, using (18) and (19), as an evolution equation for  $\delta$ ,

$$\delta'' + \frac{1}{\eta}\delta' - \frac{4}{\eta^2}\delta = -2\kappa A_2 \eta^{-1/2}, \quad (20)$$

the general solution of which is

$$\delta = \lambda_0 \eta^2 + \frac{4e_0}{\eta^2} + \frac{8}{7} \kappa A_2 \eta^{3/2}, \quad (21)$$

where  $\lambda_0$  and  $e_0$  are two constants of integration (that is two slowly varying functions of space). It is usual to call the  $\lambda_0$  term the “growing” mode [4]. Then the continuity equation (10) yields

$$h = -\frac{3}{2} \lambda_0 \eta^2 - 6 \frac{e_0}{\eta^2} - \frac{12}{7} \kappa A_2 \eta^{3/2} - 9 \Psi_0, \quad (22)$$

$\Psi_0$  being a new function of integration. The Euler equation (11) then gives  $v$ ,

$$v = -\frac{\lambda_0}{12} \eta^3 + \frac{e_0}{\eta} - \frac{4}{35} \kappa A_2 \eta^{5/2} + \lambda_1, \quad (23)$$

$\lambda_1$  being another function of integration.

At this stage of the resolution we are left with the problem of determining the second scalar perturbation of the metric (either  $h^-$  or  $E$ ) with one of the three remaining equations (12), (13), (15). For instance, the resolution of (15) yields

$$E = -e_1 - e_0 \ln \eta + \frac{1}{80} \lambda_0 \eta^4 - \frac{\Psi_0}{4} \eta^2 - \frac{\Psi_1}{\eta} + \frac{4}{441} \kappa (7A_4 - 2A_2) \eta^{7/2}, \quad (24)$$

$e_1$  and  $\Psi_1$  being yet two other integration functions.

On the six integration functions that have been introduced, namely  $\lambda_0, e_0, \Psi_0, \lambda_1, e_1$ , and  $\Psi_1$ , two of them ( $e_0$  and  $e_1$ ) can be eliminated by a gauge transformation (see eq. 3). This means that the two remaining Einstein equations (12) and (13) are not identically satisfied, but, rather, should give two constraints on the remaining four integration constants, leaving out the two physical degrees of freedom. This task requires however some shrewdness since the gradient terms in (12) and (13) cannot be simply disposed of.

In [3] another route was therefore pursued. First a “conserved pseudo-tensor”  $\tau_{\mu\nu}$  defined by [11]

$$\kappa \tau_{00} \equiv \kappa (\delta T_{00} + \Theta_{00}) - \mathcal{H} h', \quad \kappa \tau_{0k} \equiv \kappa (\delta T_{0k} + \Theta_{0k}), \quad \kappa \tau_{kl} \equiv \kappa (\delta T_{kl} + \Theta_{kl}) - \mathcal{H} (h'_{kl} - h' \delta_{kl}) \quad (25)$$

was introduced. This quantity is indeed conserved, in that  $\tau'_{0i} = \partial_j \tau_{ij}$  and  $\tau'_{00} = \partial_j \tau_{0j}$  provided Einstein’s equations (8-17) are satisfied. It has then been widely used (see e.g. [3], [4], [5], [6]) to fix the initial conditions, that is to determine the value of some of the arbitrary functions of space previously introduced. The line of the argument is that since the defect network is uncorrelated on scales larger than the Hubble radius,  $\tau_{kl}$  must be white noise on large scales, that is that its components, like the source functions  $A_i$ , must be functions of space varying on scales larger than the Hubble radius. Now, from the conservation laws one gets that  $\tau''_{00} = \partial_{ij} \tau_{ij}$ . Hence, if  $\tau_{00}$  and  $\tau'_{00}$  are continuous through the phase transition, then  $\tau_{00}$  is second order in the gradients, i.e.

$$\tau_{00} = 0 \quad (26)$$

on large scales. Now, from the definition (25) and the expressions (6), (7) for  $\delta T_{\mu\nu}$  and  $\Theta_{\mu\nu}$ , we have

$$\tau_{00} = \frac{3}{\eta^2}\delta - \frac{6A_2}{\sqrt{\eta}} - \frac{h'}{\eta} = 6\lambda_0 \quad (27)$$

(using the solution (21) and (22) for  $\delta$  and  $h$ ). Therefore the integration function  $\lambda_0$  is second order in the gradients and must be set equal to zero. Thus the use of the “conserved pseudo-tensor” (25) eliminates the “growing” mode of the density contrast  $\delta$ .

As for the other integration functions  $\Psi_0$ ,  $\Psi_1$  and  $\lambda_1$  entering the solutions (22), (23) and (24) for  $h$ ,  $v$  and  $E$  ( $e_0$  and  $e_1$  just reflect the non-unicity of synchronous gauges) they are usually [3] chosen by assuming continuity through the phase transition, that is set equal to zero.

### 3.2 Matching conditions in synchronous gauges

The choices made above to fix the functions of integration (that is to impose the continuity of  $\tau_{00}$ ,  $h$  and  $v$  and  $E$ ) must be justified. The question is therefore to determine which quantities are indeed continuous through the phase transition. This has been studied in detail in [1]. For perturbations varying on scales larger than the Hubble radius the phase transition which is at their origin looks instantaneous and the matching conditions between the pre- and post- transition eras are the standard ones : the induced three-metric on, and the extrinsic curvature of the surface of transition, which is taken to be a surface of constant density, must be continuous.

In synchronous gauges, these matching conditions translate as [1] (we give them for all  $K$ )

$$\left[ h^- + \frac{2}{3(\mathcal{H}^2 + K)(1 + \omega)} (\kappa a^2 \rho \delta + \kappa \rho^s) \right]_{\pm} = 0 \quad (28)$$

$$\left[ \mathcal{H}E' + \frac{1}{9(\mathcal{H}^2 + K)(1 + \omega)} (\kappa a^2 \rho \delta + \kappa \rho^s) \right]_{\pm} = 0 \quad (29)$$

$$\left[ \mathcal{H}h'^- - \left( 1 - \frac{2}{3} \frac{K}{(\mathcal{H}^2 + K)(1 + \omega)} \right) (\kappa a^2 \rho \delta + \kappa \rho^s) \right]_{\pm} = 0 \quad (30)$$

where  $[F]_{\pm} \equiv \lim_{\epsilon \rightarrow 0^+} (F(\eta_{PT} + \epsilon) - F(\eta_{PT} - \epsilon))$ ,  $\eta_{PT}$  being the conformal time at which the transition occurs. Two useful linear combinations will turn out to be

$$\left[ \left( 1 - \frac{2}{9} \frac{1}{(\mathcal{H}^2 + K)(1 + \omega)} \Delta \right) (\kappa a^2 \rho \delta + \kappa \rho^s) - \mathcal{H}h' + Kh^- \right]_{\pm} = 0 \quad (31)$$

$$\left[ h^- - 6\mathcal{H}E' \right]_{\pm} = 0. \quad (32)$$

### 3.3 Pseudo-tensor and matching conditions

Let us now study the implications of these general matching conditions when applied to our toy model.

First, equation (31), when particularized to a radiation background, large scale fluctuations and flat spatial sections, is nothing else than the condition of continuity of  $\tau_{00}$ . This justifies the assumption (26) and hence the choice  $\lambda_0 = 0$  (see (27)).

In order now to assess the validity of the other choices made (continuity of  $h$ ,  $v$  and  $E$ ), we shall turn to the newtonian gauge. Indeed, (32) implies, in our toy model, that

$$\left[ h^- - \frac{6}{\eta} E' \right]_{\pm} = 0 \quad (33)$$

and thus that, see (16)

$$h^- = \mathcal{O}(E), \quad h = \mathcal{O}(h^-). \quad (34)$$

A natural set of variables to solve Einstein's equations (8-15) is then  $(h^-, E)$  or  $(h, E)$ , but not  $((h - h^-), h)$ . Now, it also makes sense to find a combination of the perturbations variables such that the set of equations (10-15) reduces to a master equation and to five algebraic equations, two of which being truly redundant, and not, as is the case in synchronous gauges, constraints on integration constants. Such a combination is known and amounts to working in the newtonian gauge [2]. Indeed, defining

$$\Psi \equiv -\frac{h^-}{6} + \mathcal{H}E', \quad \Phi \equiv -E'' - \mathcal{H}E', \quad \delta^{\sharp} \equiv \delta + 3\mathcal{H}(1 + \omega)E', \quad v^{\sharp} \equiv v + E', \quad (35)$$

(with  $\mathcal{H} = 1/\eta$  and  $\omega = 1/3$  for our toy model) it is easy to show that (10-15) reduce to (see e.g. [1]) :

$$\delta^{\sharp'} = -\frac{4}{3}\Delta v^{\sharp} + 4\Psi' \quad (36)$$

$$v^{\sharp'} + \Phi + \frac{1}{4}\delta^{\sharp} = 0. \quad (37)$$

$$\Psi - \Phi = \kappa\Pi^s \quad (38)$$

$$\Delta\Psi = \frac{3}{2\eta^2} \left( \delta^{\sharp} - \frac{4}{\eta}v^{\sharp} \right) + \frac{\kappa}{2} \left( \rho^s - \frac{3}{\eta}v^s \right) \quad (39)$$

$$\Psi' + \frac{1}{\eta}\Phi = -\frac{2}{\eta^2}v^{\sharp} - \frac{\kappa}{2}v^s \quad (40)$$

$$\Psi'' + \frac{4}{\eta}\Psi' - \frac{1}{3}\Delta\Psi = \kappa \left( \frac{1}{3}\Delta\Pi^s + \frac{\Pi^{s'}}{\eta} + \frac{1}{2}P^s - \frac{1}{6}\rho^s \right). \quad (41)$$

The sources being known, the integration of (41) gives  $\Psi$  in function of two integration constants. Then  $\Phi$ ,  $v^{\sharp}$ , and  $\delta^{\sharp}$  follow algebraically from (38), (40), and (39), that is without introducing new constants. More precisely the solution of this system is (see [1])

$$\Psi = \Psi_0 + \frac{\Psi_1}{\eta^3} + \frac{2}{9}\kappa\eta^{3/2}(A_4 + A_2), \quad \Phi = \Psi_0 + \frac{\Psi_1}{\eta^3} + \frac{1}{9}\kappa\eta^{3/2}(2A_2 - 7A_4), \quad (42)$$

$$\delta^{\sharp} = -2\Psi_0 + 4\frac{\Psi_1}{\eta^3} + \frac{8}{9}\kappa\eta^{3/2}(A_2 + A_4), \quad v^{\sharp} = -\frac{1}{2}\eta\Psi_0 + \frac{\Psi_1}{\eta^2} - \frac{2}{45}\kappa\eta^{5/2}(4A_2 - 5A_4). \quad (43)$$

Only two integration “constants”,  $\Psi_0$  and  $\Psi_1$ , are introduced (and not six as is the case when working in synchronous gauges), so that the conservation equations (36) and (37)

are identically satisfied. Note that in the newtonian gauge, the “growing” mode is the term proportional to  $\Psi_0$  (and is *not* the same as the synchronous growing mode which is proportional to  $\lambda_0$ , see (21)).

Now the relations (35) can be inverted to compute  $h, h^-, \delta$  and  $v$ . Since one has to solve two differential equations to do so, two functions of space appear, which can be identified with  $e_0$  and  $e_1$ , and the comparison of (42) and (43) to (21-24) tells us that

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_0 = 0, \quad (44)$$

at lowest order in the gradients, which justifies the assumption of continuity of  $v$ , and, again,  $\tau_{00}$  ( $\tau_{00} = 0$  for large scales being nothing more than a rewriting of (39)).

Finally, the two constants  $\Psi_0$  and  $\Psi_1$  correspond to the two physical degrees of freedom and are determined by the matching conditions. They turn out not to be zero, but the terms proportional to them in (42), (43) become nevertheless quickly negligible (see [1]). Hence the assumptions that  $h$  and  $E$  be continuous are therefore asymptotically valid.

To conclude, working in synchronous gauges introduces additional integration functions which render the resolution of the perturbation equations slightly cumbersome. A solution to get rid of these spurious functions is to go to the newtonian gauge where the perturbation variables are such that the set of equations (10-15) reduces to a unique master equation and algebraic equations, two of which are redundant if the others are satisfied. The matching conditions at the surface of transition then fixes the two physical functions and justify the use of  $\tau_{00}$  to eliminate the synchronous “growing mode” (i.e.  $\lambda_0$ ), and the assumptions of continuity of  $v$  and (approximately)  $h$  and  $E$ . Now, no gauge is perfect : when the sources are incoherent,  $\Pi^s$  in the right-hand-side of (41) is a non local term.

## 4 Generalization to universes with hyperbolic sections

We now turn to universes with hyperbolic hypersurfaces ( $K = -1$ ). We will first work in the newtonian gauge and then generalize the concept of “pseudo-tensor”.

When the spatial sections are hyperbolic, the curvature generates a new length scale on top of the comoving Hubble or horizon scale ( $R_H = 1/\mathcal{H}$ ), which is the comoving curvature scale ( $R_C = 1$ ). As follows from (17), the curvature scale is larger than the horizon scale

$$R_H < R_C. \quad (45)$$

We are interested in all scales  $L$  that were larger than the horizon scale at the phase transition. They fall in three categories : (I) those which entered the horizon between the phase transition and today (i.e.  $L \leq R_{H_0}$ ), (II) those which have not yet entered the horizon but are subcurvature (i.e.  $R_{H_0} \leq L \leq R_C$ ), (III) those which are supercurvature (i.e.  $L > R_C$ ).

### 4.1 Newtonian gauge

As we have seen, the introduction of the newtonian gauge perturbations (35) was useful to keep to a minimum the number of integration functions. The generalization to  $K \neq 0$

spacetimes governed by a mixture of fluids of the master equation for  $\Psi$  (41) is (see e.g. [8])

$$\Psi'' + 3(1 + c_s^2)\mathcal{H}\Psi' + [2\mathcal{H}' + (\mathcal{H}^2 - K)(1 + 3c_s^2)]\Psi - c_s^2\Delta\Psi = \mathcal{S} \quad (46)$$

where  $\mathcal{S}$  is a source involving the variables describing the topological defects, whether they satisfy (19) or not. Equation (46) is closed, in the sense that  $\mathcal{S}$  is a known function of time, only in the case when the background is governed by a single fluid. In the general case of a multi-fluid system, one has to add a set of equations for the relative entropies which enter  $\mathcal{S}$  (see [8]). However, on super-horizon scales, all these entropies turn out to be constant so that the multi-fluid can be described as a single one and the equation is again closed. In this paragraph, we therefore restrict our attention to modes which have not yet entered the horizon that is such that ( $L > R_{H_0}$ ), i.e. from categories II and III. For those modes, it is easy to show that the term  $c_s^2\Delta\Psi$  is negligible compared to  $(\mathcal{H}^2 - K)\Psi$  in the regime when  $\mathcal{H}^2$  is of order  $K$  (see Appendix 1).

Equation (46) can then be rewritten under quite a compact form as

$$\chi' = \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}' + K}\mathcal{S}, \quad (47)$$

where we have introduced instead of  $\Psi$  the new variable  $\chi$  defined by

$$\chi \equiv \frac{\mathcal{H}^2}{\mathcal{H}^2 - \mathcal{H}' + K} \frac{1}{a^2} \left( \frac{\Psi a^2}{\mathcal{H}} \right)'. \quad (48)$$

The general solution of (47) can be written formally as

$$\chi = \chi_0 + \int^\eta d\bar{\eta} \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}' + K} \mathcal{S} \quad (49)$$

where  $\chi_0$  is an integration constant, that is a slowly varying function of space only. Inverting equation (48), we can therefore write the general solution of equation (46) on superhorizon scales as

$$\frac{a^2}{\mathcal{H}}\Psi = \chi_1 - \chi_0 \int^\eta d\bar{\eta} a^2 \frac{\mathcal{H}^2 - \mathcal{H}' + K}{\mathcal{H}^2} - \int^\eta d\bar{\eta} a^2 \frac{\mathcal{H}^2 - \mathcal{H}' + K}{\mathcal{H}^2} \int^{\bar{\eta}} d\tilde{\eta} \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}' + K} \mathcal{S} \quad (50)$$

where  $\chi_1$  is another integration function.  $\chi_0$  and  $\chi_1$  are determined by the matching conditions as in the flat case (in which case they are proportional to  $\Psi_0$  and  $\Psi_1$ ).

(Note that in the absence of the source term  $\mathcal{S}$ , as is the case in inflationary models,  $\chi$  generalizes Lyth's variable  $\zeta$  [14] to Friedmann-Lemaître spacetimes with hyperbolic spatial sections. Its constancy on superhorizon scales (equation (49)) can then be used to compute, by means of (50), the amplification of  $\Psi$  in open inflationary models.)

As for the other perturbations, they are given algebraically by

$$\begin{aligned} \Phi &= \Psi - \kappa\Pi^s, \quad v^\sharp = -\frac{2}{3(\mathcal{H}^2 + K)(1 + \omega)} \left( \Psi' + \mathcal{H}\Phi + \frac{\kappa}{2}v^s \right), \\ \delta^\sharp &= \frac{1}{3(\mathcal{H}^2 + K)} \left\{ \kappa(\rho^2 - 3\mathcal{H}v^s) - 6\mathcal{H}(\Psi' + \mathcal{H}\Phi) \right\}. \end{aligned} \quad (51)$$

(which generalize equations (38) (39) (40) (41)). The solution depends only on the two integration functions  $\chi_0$  and  $\chi_1$ , and the continuity and the Euler equations are identically satisfied.

## 4.2 Synchronous gauges

Provided we have solved the equations of the background (17), the set of equations (10, 11, 14) can be solved as in the section 3.1. However, the question is then to find a prescription to fix the coefficient of the synchronous “growing mode” of the density contrast (i.e. the equivalent of the constant  $\lambda_0$  in (21)). For that purpose, we will generalize  $\tau_{00}$ .

If we note that, when  $K = 0$ ,  $\tau_{00} = 0$ ,  $\tau_{00}$  being given by (25), is in fact a way to write the matching condition (31) at lowest order, a straightforward generalization of  $\tau_{00}$  is

$$\kappa\tau_{00} \equiv \sqrt{\gamma} \left\{ \kappa\delta T_{00} + \kappa\Theta_{00} + Kh^- - \mathcal{H}h' \right\} \quad (52)$$

where  $\gamma$  is the determinant of the 3-metric  $\gamma_{ij}$ . Indeed, if we introduce

$$\kappa\tau_{0k} \equiv \sqrt{\gamma} \left\{ \kappa\delta T_{0k} - 2K\partial_k E' \right\}, \quad (53)$$

it can easily be checked, using the Einstein equations (12, 13), that  $\tau_{0\nu}$  is conserved, in the sense that

$$\partial_0\tau_{00} = \partial_k\tau_{0k}. \quad (54)$$

Now, the matching condition (31) imposes that initially

$$\tau_{00} = 0 \quad (55)$$

at lowest order on superhorizon scales. Furthermore (54) implies that  $\tau_{00}$  remains zero at all times for superhorizon modes. Thus (55) is a way to eliminate at all times the spurious growing mode in the solution for the density contrast  $\delta$ .

## 5 The matching conditions in terms of a vector conservation law

In the previous section we have encoded part of the matching conditions into a quantity  $\tau_{00}$ , the geometrical status of which is not defined. We show here that for long wavelength modes, it can be interpreted in terms of a conserved vector related to a symmetry of the background. For that purpose, we briefly describe the formalism we use and then apply it to our problem.

### 5.1 General Formalism

A general formalism was developped by Katz-Bičák-Lynden Bell [7] to define conserved quantities and conservation laws in an arbitrary spacetime  $(\mathcal{M}, g_{\mu\nu})$ . This generalizes the work by Bergmann who defined conservation laws with respect to a flat background [15]. The formalism involves the introduction of a background  $(\bar{\mathcal{M}}, \bar{g}_{\mu\nu})$  and a mapping, i.e. a way to identify points of  $\mathcal{M}$  and  $\bar{\mathcal{M}}$ .

The central idea is that one can construct a lagrangian for the gravitational field, quadratic in the covariant derivatives of the metrics and normalized so that it vanishes on the background (i.e. when  $\mathcal{M} = \bar{\mathcal{M}}$ ), which is a true scalar density. From this quantity, one can therefore build vectors densities  $\hat{I}^\mu$ , which are conserved in the sense that  $\partial_\mu \hat{I}^\mu = 0$ .

The detailed construction of the vector densities  $\hat{I}^\mu$  and the demonstration of their properties can be found in [7]. We will just sum up the relations which will be useful and fix the notations. The conserved vector field density  $\hat{I}^\mu$  is given by

$$\kappa \hat{I}^\mu \equiv \left[ \left( \hat{G}_\nu^\mu - \bar{G}_\nu^\mu \right) + \frac{1}{2} \hat{l}^{\rho\sigma} \bar{R}_{\rho\sigma} \delta_\nu^\mu + \kappa \hat{t}_\nu^\mu \right] \xi^\nu + \kappa \hat{\sigma}^{\mu[\rho\sigma]} \partial_{[\rho} \xi_{\sigma]} + \kappa \hat{Z}^\mu(\xi^\nu), \quad (56)$$

where  $\xi^\mu$  is an arbitrary vector field. For any quantity  $A$ ,  $\hat{A}$  denotes  $\sqrt{-g}A$ ,  $g$  being the determinant of the metric  $g_{\mu\nu}$ , and  $\bar{A}$  the value of  $A$  on the background.  $\hat{G}_\nu^\mu$  and  $\bar{G}_\nu^\mu$  are the Einstein tensor densities of the spacetime and of the background,  $\hat{l}^{\mu\nu}$  is the difference of the two metric tensor densities,

$$\hat{l}^{\mu\nu} \equiv \hat{g}^{\mu\nu} - \bar{g}^{\mu\nu},$$

$\bar{R}_{\mu\nu}$  is the Ricci tensor of the background spacetime. Introducing

$$\Delta_{\mu\nu}^\lambda \equiv \Gamma_{\mu\nu}^\lambda - \bar{\Gamma}_{\mu\nu}^\lambda,$$

where  $\Gamma_{\mu\nu}^\lambda$  and  $\bar{\Gamma}_{\mu\nu}^\lambda$  are the Christoffel symbols of the spacetime and of the background, as well as  $\bar{\nabla}_\mu$  and  $\nabla_\mu$  the two covariant derivatives associated with  $\bar{g}_{\mu\nu}$  and  $g_{\mu\nu}$ , the expressions for  $\hat{t}_\nu^\mu$ ,  $\hat{\sigma}^{\mu[\rho\sigma]}$  and  $\hat{Z}^\mu(\xi^\nu)$  are

$$\begin{aligned} 2\kappa \hat{t}_\nu^\mu &= \hat{g}^{\rho\sigma} \left( \Delta_{\rho\lambda}^\lambda \Delta_{\sigma\nu}^\mu + \Delta_{\rho\sigma}^\mu \Delta_{\lambda\nu}^\lambda - 2\Delta_{\rho\lambda}^\mu \Delta_{\sigma\nu}^\lambda \right) \\ &\quad + \hat{g}^{\mu\rho} \left( \Delta_{\lambda\sigma}^\sigma \Delta_{\rho\nu}^\lambda - \Delta_{\sigma\rho}^\sigma \Delta_{\lambda\nu}^\lambda \right) - \hat{g}^{\rho\sigma} \left( \Delta_{\rho\sigma}^\eta \Delta_{\lambda\eta}^\lambda - \Delta_{\rho\lambda}^\eta \Delta_{\sigma\eta}^\lambda \right) \delta_\nu^\mu \end{aligned} \quad (57)$$

(this term reduces to the Einstein pseudo-tensor density when the background is Minkowski spacetime in cartesian coordinates),

$$2\kappa \hat{\sigma}^{\mu[\rho\sigma]} = \left( \hat{l}^{\mu[\rho} \bar{g}^{\sigma]\lambda} - \bar{g}^{\mu[\rho} \hat{l}^{\sigma]\lambda} \right) \Delta_{\lambda\nu}^\nu - 2\hat{l}^{\lambda[\rho} \bar{g}^{\sigma]\nu} \Delta_{\lambda\nu}^\mu \quad (58)$$

(where  $\square$  means antisymmetrization), and

$$\begin{aligned} 4\kappa \hat{Z}^\mu(\xi^\nu) &= \left( Z_\rho^\mu \hat{g}^{\rho\sigma} + \hat{g}^{\mu\rho} Z_\rho^\sigma - \hat{g}^{\mu\sigma} Z \right) \Delta_{\sigma\lambda}^\lambda \\ &\quad + \left( \hat{g}^{\rho\sigma} Z - 2\hat{g}^{\rho\lambda} Z_\lambda^\sigma \right) \Delta_{\rho\sigma}^\mu + \hat{g}^{\mu\lambda} \partial_\lambda Z + \hat{g}^{\rho\sigma} \left( \bar{\nabla}^\mu Z_{\rho\sigma} - 2\bar{\nabla}_\rho Z_\sigma^\mu \right), \end{aligned} \quad (59)$$

where  $Z_{\rho\sigma}$  and  $Z$  are given by

$$Z_{\rho\sigma} = \mathcal{L}_\xi \bar{g}_{\rho\sigma} = 2\bar{\nabla}_{(\rho} \xi_{\sigma)} \quad \text{and} \quad Z = Z_{\rho\sigma} \bar{g}^{\rho\sigma} \quad (60)$$

(where  $()$  means symmetrization).

Let us emphasize that the relation

$$\partial_\mu \hat{I}^\mu = 0 \quad (61)$$

is an identity which holds for all  $(g_{\mu\nu}, \bar{g}_{\mu\nu}, \xi^\mu)$ . The vector field  $\xi^\mu$  can be seen as generating an infinitesimal change of coordinates : contrarily to special relativity, one can build an infinite numbers of conserved vectors corresponding to all reparametrizations. These conserved vectors are useful to write integral conservation laws (known as “strong conservation laws”). When  $\xi^\mu$  is a Killing vector field of the background, these strong conservation laws are called “Noether conservation laws”, i.e. conservation laws related to a symmetry.

We also stress that up to now, we have done no linearization; the case when  $\mathcal{M}$  is a “small perturbation” of  $\bar{\mathcal{M}}$  is considered in the next section.

## 5.2 Conserved vectors in a perturbed Friedmann-Lemaître space-time

We now apply this general formalism to perturbations in cosmology. The manifold  $\mathcal{M}$  is a perturbed Friedmann-Lemaître universe and the background is taken to be a Friedmann-Lemaître spacetime, as it is usual in the theory of cosmological perturbations. Let us note that other choices for the background can be made, for instance de Sitter spacetime (see e.g. [16]). The line element of  $\mathcal{M}$  can then be written as

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = (\bar{g}_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu, \quad (62)$$

where  $h_{\mu\nu}$  is a “small perturbation”.

We can now linearize equation (56). A first step yields [7]

$$\kappa \hat{I}^\mu = \sqrt{-\bar{g}} \left[ \delta G_\nu^\mu + \frac{1}{2} \left( \bar{R}_\nu^\mu \delta_\rho^\sigma - \bar{R}_\rho^\sigma \delta_\nu^\mu \right) h_\sigma^\rho \right] \xi^\nu + \kappa \hat{\mathcal{Z}}^\mu + \mathcal{O}(h^2). \quad (63)$$

The second step is to linearize  $\mathcal{Z}^\mu$ . To do that we need to specify the arbitrary vector field  $\xi^\mu$ . As discussed before the relation  $\partial_\mu \hat{I}^\mu = 0$  is valid whatever field we use. However Killing vectors of the background (such that  $\bar{\nabla}_{(\mu} \xi_{\nu)} = 0$ ) are of special interest since they are related to symmetries and have a geometrical interpretation. Unfortunately the six Killing vectors of the Friedmann-Lemaître background lay in spacelike hypersurfaces and cannot generate any quantity related to time translation, that is to energy. Now, we know that when two spacetimes are conformal, the Killing vectors of one of the spacetimes are at least conformal Killing vectors (that is such that  $\bar{\nabla}_{(\mu} \xi_{\nu)} = \frac{1}{4} \bar{g}_{\mu\nu} \bar{\nabla}_\rho \xi^\rho$ ) of the other [17]. We also know that the Friedmann-Lemaître spacetimes are conformal to Einstein, Minkowski and de Sitter spacetimes [18], each of which has a Killing vector field associated with time translation that induces a conformal Killing vector field on the Friedmann-Lemaître spacetimes. In the following, we shall use one particular conformal Killing vector to generalize the symmetry under time translation. Let us first construct this vector field explicitly.

The background metric being written as

$$d\bar{s}^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) \left[ -d\eta^2 + \gamma_{ij} dx^i dx^j \right] \quad \text{with} \quad \gamma_{ij} = \delta_{ij} + K \frac{\delta_{im} \delta_{jn} x^m x^n}{1 - K \delta_{mn} x^m x^n} \quad (64)$$

where  $\gamma_{ij}$  is the spatial metric in Weinberg’s coordinates [19], we choose the conformal Killing vector defined by

$$\xi^\mu = \delta_0^\mu, \quad (65)$$

which is a Killing vector of Einstein static spacetimes, i.e. obtained by setting  $a = 1$  in (64). This vector is associated with translations in cosmic time, i.e. in the proper time of a comoving observer of the Friedmann-Lemaître background. Other conformal Killing vectors could have been used (for instance the one associated with the conformity to de Sitter spacetime, see e.g. [16]), but (65) is the useful one for the law we want to generalize.

The vector  $\xi^\mu$  having thus being chosen, we can now linearize  $\mathcal{Z}^\mu$ . We see from equation (60) that  $\mathcal{Z}^\mu$  vanishes for any Killing vector field. For any conformal Killing vector field equation (59) gives after linearization [7],

$$8\kappa \mathcal{Z}^\mu = (h\bar{g}^{\mu\rho} - h^{\mu\rho}) \partial_\rho \mathcal{Z} - \mathcal{Z} \bar{\nabla}_\rho (h\bar{g}^{\mu\rho} - h^{\mu\rho}), \quad (66)$$

with  $h = h_{\mu\nu}\bar{g}^{\mu\nu}$  and  $Z$  given by equation (60).

Let us now, as a third step, specialize to scalar perturbations, that is to the sub-class of perturbations in synchronous gauges as defined in (1). We now have all the elements to compute the conserved vector associated to the conformal Killing vector field (65). One can first easily show that

$$Z_{\mu\nu} = 2\mathcal{H}g_{\mu\nu} \quad \text{and} \quad Z = 8\mathcal{H}, \quad (67)$$

and a straightforward calculation then yields :

$$\begin{aligned} \kappa\hat{I}^0 &= a^4\sqrt{\gamma}\left[\kappa\delta T_0^0 + \kappa\Theta_0^0 - \frac{K}{a^2}h + \frac{\mathcal{H}}{a^2}h'\right] \\ \kappa\hat{I}^k &= a^4\sqrt{\gamma}\left[\kappa\delta T_0^k + \kappa\Theta_0^k + \frac{\mathcal{H}}{a^2}(D^l h_l^k - D^k h)\right], \end{aligned} \quad (68)$$

In synchronous gauge (68) can be written in terms of  $E$  and  $h$  as

$$\begin{aligned} \kappa\hat{I}^0 &= -a^2\sqrt{\gamma}[\kappa(\delta T_{00} + \Theta_{00}) + Kh - \mathcal{H}h'] \\ \kappa\hat{I}^k &= a^2\sqrt{\gamma}\left[\kappa(\delta T_{0k} + \Theta_{0k}) + 2\mathcal{H}\gamma^{kl}\partial_l\left(E - \frac{1}{3}h^-\right)\right], \end{aligned} \quad (69)$$

It is clear that the identity (61) with  $\hat{I}^\mu$  given by (68) or (69) is a consequence of the Einstein constraint equations (12-13), but a useful one as we shall see.

### 5.3 Encoding the matching conditions in a conserved vector field

Comparing the expression of  $\tau_{00}$  (52) and  $\tau_{0k}$  (53) to  $\hat{I}^\mu$  (69), and recalling that  $h = h^-$  on superhorizon scales, we have that, in that limit :

$$\tau_{00} = -\frac{\hat{I}^0}{a^2}, \quad \partial_k\tau_{0k} = \frac{1}{a^2}\left[\partial_k\hat{I}^k + 2\mathcal{H}\hat{I}^0\right]. \quad (70)$$

Thus, for superhorizon scales, the matching condition (31) can be encoded in the components of a conserved vector field related to a symmetry since

$$[\tau_{00}]_\pm = 0 \quad \Longleftrightarrow \quad [\hat{I}^0]_\pm = 0. \quad (71)$$

This enables us to relate  $\tau_{\mu 0}$ , in the long wavelength limit, to a vector field built with the conformal Killing vector field  $\xi^\mu$  associated with time translation. In fact the construction of  $\tau_{\mu\nu}$  uses in an obvious way the fact that a Friedmann-Lemaître universe with flat spatial sections is conformal to Minkowski spacetime (see Verraraghavan and Stebbins [11] section VI and Pen et al. [3] section IV-F).

$\tau_{00}$  is often referred to as a “pseudo-energy” [4]. Our construction can justify such a denomination since it is related to a symmetry involving time translation.

## Aknowledgments

JPU and ND thanks J. Katz, D. Langlois, P. Peter and A. Riazuelo for many enlightening discussions and the programme Alliance who enabled them to visit DAMTP. We are grateful to our referee whose remarks greatly helped to improve the presentation of this paper.

# Appendix 1

We compare the terms  $c_s^2 \Delta \Psi$  and  $(\mathcal{H}^2 - K)(1 + 3c_s^2)\Psi$  in equation (46) for modes of characteristic length  $L$  (i.e. such that  $\Delta \Psi \sim \Psi/L^2$ ) in a universe with hyperbolic spatial sections ( $K = -1$ ) and dominated by a mixture of radiation (subscript  $r$ ) and dust (subscript  $m$ ).

Using the fact that  $P = P_r$  and  $\rho = \rho_m + \rho_r$ , we can relate the sound speed  $c_s^2 \equiv P'/\rho'$  to the energy density of the radiation and the dust by

$$c_s^2 = \frac{1}{3} \frac{1}{1 + \frac{3}{4} \frac{\rho_m}{\rho_r}}. \quad (72)$$

If we now introduce  $x \equiv a/a_0$  and  $\Omega_0^X \equiv \kappa \rho_0^X / 3H_0^2$  (where a subscript 0 means that we evaluate the quantity today and  $X$  stands for either  $r$  or  $m$ ), and use the Friedmann equation (17), we have that

$$c_s^2 \Delta \Psi \ll (\mathcal{H}^2 - K)(1 + 3c_s^2)\psi \iff L^{-2} \ll 3 \left( 1 + \frac{3}{4} x \frac{\Omega_0^m}{\Omega_0^r} \right) \left( 2 + \frac{\Omega_0^m}{1 - \Omega_0} \frac{1}{x^2} \left[ x + \frac{\Omega_0^r}{\Omega_0^m} \right] \right). \quad (73)$$

All modes belonging to the categories (II) and (III) are such that

$$L^{-1} < R_{H_0}^{-1}. \quad (74)$$

Therefore, the inequality (73) is satisfied for all  $x$  such that

$$1 + \frac{\Omega_0^r}{1 - \Omega_0} \ll 3 \left( 1 + \frac{3}{4} x \frac{\Omega_0^m}{\Omega_0^r} \right) \left( 2 + \frac{\Omega_0^m}{1 - \Omega_0} \frac{1}{x^2} \left[ x + \frac{\Omega_0^r}{\Omega_0^m} \right] \right). \quad (75)$$

For  $\Omega_0^m \sim 0.2$ , it is easy to see that this inequality is satisfied for all  $x$  in  $]0, 1]$ .

## References

- [1] Deruelle N., Langlois D. and Uzan J.-P., Phys. Rev. **D56**, 7608 (1998).
- [2] J.M. Bardeen, Phys. Rev. **D22**, 1882 (1980).
- [3] U-L. Pen, D.N. Spergel and N. Turok, Phys. Rev **D49**, 693 (1994).
- [4] N. Turok, Phys. Rev. **D54**, 3686 (1996); N. Turok, Phys. Rev. Lett. **77**, 4138 (1996).
- [5] W. Hu, D.N. Spergel and M. White, Phys. Rev. **D55**, 3288 (1997).
- [6] W. Hu and M. White, Phys. Rev. **D56**, 596 (1997).
- [7] J. Katz, J. Bičák and D. Lynden Bell, Phys. Rev. **D55**, 5957 (1997).
- [8] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78** (1984) ; V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, Phys. Rep. **215**, 203 (1992) ; R. Dürer, Fund. of Cosmic Physics **15**, 209 (1994).

- [9] R. Dürer and Z.H. Zhou, Phys. Rev. **D53**, 5384 (1994).
- [10] R. Dürer and M. Sakellariadou, Phys. Rev **D56**, 4480 (1997).
- [11] S. Veeraraghavan and A. Stebbins, Ap. J. **365**, 37 (1990).
- [12] D. Langlois, Phys. Rev. **D55**, 7389 (1997).
- [13] N. Turok, astro-ph/9704165
- [14] D. Lyth and A.R. Liddle, Phys. Rep. **231**, 1 (1993).
- [15] P.G. Bergmann, Phys. Rev. **75**, 680 (1949).
- [16] N. Deruelle, J. Katz and J.P. Uzan, Class. Quant. Grav. **14**, 421 (1997).
- [17] Eisenhardt, “Riemannian geometry”, Princeton University Press (1950).
- [18] S. Hawking and G.F.R. Ellis, “The large scale structure of spacetime”, Cambridge University Press (1973).
- [19] S. Weinberg, “Gravitation and Cosmology”, Wiley (1972).